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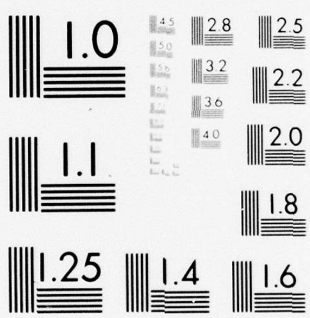

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TUNABLE INTEGRATION AND TUNABLE TRAPEZOIDAL CONVOLUTION – A POTPOURRI

U.S. ARMY MISSILE RESEARCH AND DEVELOPMENT COMMAND

Aeroballistics Directorate
Technology Laboratory

5 May 1977

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

Tunable integration and trapezoidal convolution are reviewed and tunable trapezoidal convolution derived. The effects of tuning for a sine wave input to a tunable integrator are shown analytically. Also there are some examples as to how recurrence relations may be developed using z-transforms and trapezoidal convolution.

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PREFACE

This study was initially motivated by Jon Mike Smith's seminar "Recent Developments in Modern Numerical Methods and Cost Saving Techniques" which was held at Redstone Arsenal on 12, 13, and 14 February 1976 [1]. It was further motivated by Professor Charles A. Halijak's briefing on "Numerical Transforms and Digital Simulation of Dynamical Systems" held during July and August 1976.

This report documents some of the results of the study undertaken as a consequence of the previously mentioned seminar and briefing. Though not intended to be tutorial, it is hoped that it will be comprehensible to those acquainted with z-transforms.

ACKNOWLEDGEMENTS

The author wishes to acknowledge the debt he owes Dr. Halijak and Mike Smith, respectively, for the excellent briefing and seminar. The author wishes also to acknowledge the willing ear and helpful comments of a coworker, Victor Grimes.

SYMBOLS

η	Eta, phase compensation or interpolation factor
$\mathcal{L}(\ )$	Laplace transform operator
n	Index or step number
T	Sample time or time step
$x^{(n)}$	n-th derivative of x(t)
$x^{(-n)}$	n-th integral of x(t)
z	delayer, e^{-sT}
Z( )	z-transform operator (Appendix A)

I. INTRODUCTION

The rapid development of digital computers since World War II has prompted a reexamination of numerical computation techniques from the sampled-data point of view. A very interesting outcome of this point of view is that some of the classical integrators are actually phase shifted variations of the same integrator [1,2]. For example, the integration recurrence relation

$$x_{n+1} = x_n + T\,[\eta \dot{x}_{n+1} + (1 - \eta)\dot{x}_n] \tag{1}$$

leads to the following classical integration formulas (Table 1).

TABLE 1. EQUIVALENCE OF TUNED AND CLASSICAL INTEGRATORS

Phase Shift (η)	Classical Name
0	Euler (left Riemann sum)
1/2	Trapezoidal
1	Rectangular (right Riemann sum)
3/2	Adams second order

The fact that the classical integrators listed in Table 1, differ only in their phase is quite interesting; even more interesting is the possiblity of phase shifts leading to integrators between the classical ones.

II. THE TUNABLE INTEGRATOR

The digital computer is of necessity a discrete time system and, as such, is amenable to analysis using z-transforms [3, 4, 5].

It is well known in z-transforms that

$$Z[f(s)g(s)] = y(z)f(z),\ y(s) \neq g(s) \quad . \tag{2}$$

It follows that

$$y(z) = \frac{Z[f(s)g(s)]}{f(z)} \tag{3}$$

and taking g(s) as the input and f(s) as the plant, y(z) cannot be determined without knowing the plant, f(s).

If a continuous system is to be simulated on a digital computer, approximations must be made. For example, if g(s) is not known a priori [Figure 1(a)], Z[g(s)f(s)] must be approximated in order to develop a recurrence relation for digital simulation of g(s)f(s).

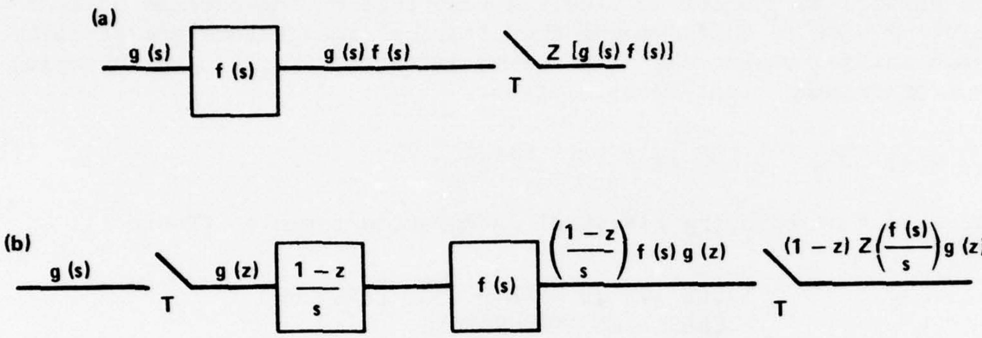

Figure 1. Continuous system.

If the input, g(s), is sampled and then held by a zero order hold [Figure 1(b)], the output after the second sampler is

$$Z\left[\frac{1 - z}{s} f(s)g(z)\right] \quad , \tag{4}$$

which may be simplified to

$$(1 - z)\ Z\left(\frac{f(s)}{s}\right)\ g(z) \quad . \tag{5}$$

Given

$$f(s) = \frac{1}{s} \quad , \tag{6}$$

the modified z-transform of $1/s^2$, substituting Equation (6) into Equation (5), is (Appendix B)

$$\frac{T[\eta + (1 - \eta)z]}{(1 - z)^2} \tag{7}$$

and Equation (5) becomes

$$Z\left(\frac{g(s)}{s}\right) \simeq \frac{T[\eta + (1 - \eta)z]\ y(z)}{(1 - z)} \quad , \tag{8}$$

the tunable integration Equation (1), in z-transform notation. The details of the conversion to a recurrence are shown in the sample problem section.

For a sufficiently fine timestep, T, it is hoped that

$$y(z) \simeq T[\eta + (1 - \eta)z]g(z) \tag{9}$$

and for a given input, g(s), Equation (3) would allow a check on the approximation, Equation (9), for f(s), an integrator.

An attempt might be made to extend this procedure further by integrating f(s) by "inserting" another sample and hold,

$$Z\left(\frac{f(s)}{s}\right) \simeq (1 - z)Z\left(\frac{1}{s^2}\right) f(z) \tag{10}$$

and, after taking the modified z-transform of $1/s^2$,

$$Z[g(s)f(s)] \approx T[\eta + (1 - \eta)z]f(z)g(z) \tag{11}$$

or

$$y(z) \approx T[\eta + (1 - \eta)z]g(z) \quad . \tag{12}$$

for any f(s).

The insertion of this sample and hold, Equation (10), is not physically realizable because f(z) is not an input; this attempted extension is not the proper approximation as will be seen in the next section.

III. TUNABLE TRAPEZOIDAL CONVOLUTION

If a substitution of the definition of the z-transform, Equation (A-10), is made into Equation (2), it becomes

$$\sum_{n=0}^{\infty} [f(nt)*g(nT)]z^n = \left[\sum_{n=0}^{\infty} y(nT)z^n\right]\left[\sum_{n=0}^{\infty} f(nT)z^n\right] \quad . \tag{13}$$

Upon substitution of the definition of convolution and taking the Gauchy product of power series, the following is obtained:

$$\sum_{n=0}^{\infty} z^n \int_0^{nT} g(\tau)f(nT - \tau)d\tau = \sum_{n=0}^{\infty} z^n \sum_{k=0}^{n} y(kt)f(nT - kT) \quad . \tag{14}$$

Equating coefficients of powers of z, the following is obtained:

$$\int_0^{nT} g(\tau)f(nT - \tau)\, d\tau = \sum_{k=0}^{n} y(kT)f(nT - kT) \quad . \tag{15}$$

The mean value theorem of the integral calculus guarantees that there is some γ_n such that [6],

$$\int_0^{nT} g(\tau)\ f(nT - \tau)d\tau = nT\ g(\gamma_n\ nT)f(nT - \gamma_n\ nT),\ 0 \le \gamma_n \le 1 \quad ; \tag{16}$$

of course,

$$\int_0^{nT} g(\tau)\ f(nT - \tau)d\tau = \sum_{k=0}^{n-1} \int_{kT}^{(k+1)T} g(\tau)f(nT - \tau)d\tau \tag{17}$$

and there is some γ_k such that

$$\int_{kT}^{(k+1)T} g(\tau)\ f(nT - \tau)d\tau = T\ g(kT + \gamma_k T)f(nT - kT - \gamma_k T), \quad 0 \le \gamma_k \le 1 \quad . \tag{18}$$

Knowing γ_n and γ_k exist is like knowing the solution to a differential equation exists; knowledge of the existence of a solution is not knowledge of the solution. Of course, it is wise to check for existence of a solution before seeking a solution.

For a sufficiently small time step, T, trapezoidal quadrature might be used to approximate Equation (18),

$$Tg(kT - \gamma_k T)f(nT - kT - \gamma_k T) \simeq \frac{T}{2}[g(kT + T)f(nT - kT - T) + g(kT)f(nT - kT)] \quad . \tag{19}$$

Then, the right hand side of Equation (16) is approximately

$$\frac{T}{2}\sum_{k=0}^{n-1}[g(kT+T)f(nT-kT-T)+g(kT)f(nT-kT)] \tag{20}$$

and changing the summing indices

$$\frac{T}{2}\sum_{k=1}^{n} g(kT)\ f(nT-kT)+\frac{T}{2}\sum_{k=0}^{n-1} g(kT)f(nT-kT) \tag{21}$$

and combining,

$$T\sum_{k=0}^{n} g(kT)f(nT-kT)-\frac{T}{2}\left[g_0 f(nT)+f_0 g(nT)\right] \quad . \tag{22}$$

It follows that, Equations (22) and (15),

$$\sum_{k=0}^{n} y(kT)f(nT-kT)\simeq T\sum_{k=0}^{n} g(nT)f(nT-kT)-\frac{T}{2}[g_0 f(nT)+f_0 g(nT)] \ . \tag{23}$$

For f_0 and g_0 both zero,

$$y(kT)\simeq T\ g(kT) \tag{24}$$

and from the definition of the z-transform it follows that

$$y(z)\simeq T\ g(z) \tag{25}$$

and

$$Z[g(s)f(s)]\simeq T\ g(z)f(z) \quad . \tag{26}$$

Unfortunately, requiring f_o and g_o both to be zero is unusually restrictive because the initial conditions on the input must be zero and many plants of interest are excluded including the single integrator (Appendix B).

Substituting Equation (23) into the right hand side of Equation (14),

$$T \sum_{n=0}^{\infty} z^n \left\{ \sum_{k=0}^{n} g(kT)f(nT - kT) - 1/2\ f_0 g(nT) - 1/2\ g_0 f(nT) \right\} \quad , \tag{27}$$

$$T \sum_{n=0}^{\infty} \sum_{k=0}^{n} g(kT)z^k\ f(nT\ kT)z^{n-k} - \frac{T}{2} \left[f_0 \sum_{n=0}^{\infty} g(nT)z^n + g_0 \sum_{n=0}^{\infty} f(nT)z^n \right] \quad , \tag{28}$$

$$T \left[\sum_{n=0}^{\infty} f(nT)z^n \right] \left[\sum_{n=0}^{\infty} g(nT)z^n \right] - \frac{T}{2} \left[f_0 \sum_{n=0}^{\infty} g(nT)z^n + g_0 \sum_{n=0}^{\infty} f(nT)z^n \right] \tag{29}$$

and, finally, from the definition of the z-transform, and Equation (29)

$$Z[f(s)g(s)] \simeq T\ f(z)g(z) - \frac{T}{2} [f_0 g(z) + g_0 f(z)] \quad , \tag{30}$$

trapezoidal convolution [7,8,4,]. It follows that

$$y(z) \simeq T \left[1 - \frac{1}{2} \frac{f_0}{f(z)} \right] g(z) - \frac{T}{2} g_0 \quad . \tag{31}$$

If time step, T, is to be made as large as possible for reasons of speed in real time simulation, or economy in Monte Carlo studies, Equation (30) may limit how large the time step, T, may become. In using tunable integration linear interpolation is being used for Equations (18),

$$\begin{aligned} Tg(kT + \gamma_k T)f(nT - kT - \gamma_k T) &\simeq T\eta_k\ g(kT + T)f(nT - kT - T) \\ &+ T(1 - \eta_k)g(kT)f(nT - kT) \quad . \end{aligned} \tag{32}$$

Of course, there is no guarantee that a linear interpolator will find Equation (18) on the interval, $0 \le \eta_k \le 1$. It may be necessary to extrapolate. Extrapolation may not produce the desired values of the functions; the values at the ends of the interval could be equal. Hopefully, this would be the exception rather than the rule.

Then Equation (17) upon substitution of Equation (32), would be

$$\int_{o}^{nT} g(\tau)\ f(nT - \tau)d\tau \simeq T \sum_{k=0}^{n-1} \Big[\eta_{k+1}\ g(kT + T)f(nT - kT - T) + (1 - \eta_{k+1})\ g(kT)f(nT - kT) \Big] \tag{33}$$

$$\simeq T \sum_{k=1}^{n} \eta_k\ g(kT)f(nT - kT) + T \sum_{k=0}^{n-1} (1 - \eta_{k+1})\ g(kT)f(nT - kT) \tag{34}$$

$$\simeq T \sum_{k=0}^{n} g(kT)f(nT - kT) + T \sum_{k=1}^{n-1} (\eta_k - \eta_{k+1})g(kT)f(nT - kT) - T[\eta_o\ g_o f(nT) + (1 - \eta_n)f_o g(nT)] \tag{35}$$

Assuming

$$\eta_k \simeq \eta_{k+1} \tag{36}$$

Equation (35) may be written

$$\simeq T \sum_{k=0}^{n} g(kT)f(nT\ kT) - T[\eta_o\ g_o\ f(nT) + (1 - \eta_n)f_o g(nT)] \tag{37}$$

and it follows that

$$Z[f(s)g(s)] \simeq Tg(z)f(z) - T[\eta_o g_o f(z) + (1 - \eta_\infty)f_o g(z)] \tag{38}$$

Making a final assumption,

$$\eta_o \simeq \eta_\infty \tag{39}$$

the following is obtained:

$$Z[f(s)g(s)] \simeq Tg(z)f(z) - T[\eta g_o f(z) + (1 - \eta)f_o g(z)] \tag{40}$$

tunable trapezoidal convolution. The final assumption, Equation (39), does not appear reasonable, but the conditions under which it may be justified will be discussed in the analysis of tuning. From Equation (40) it follows that

$$y(z) \simeq T\left[1 - (1-\eta)\frac{f_o}{f(z)}\right] g(z) - T\eta g_o \quad . \tag{41}$$

In Section II, a digital-to-analog converter was used to reconstruct an analog representation of the input [Figure 2(a)].

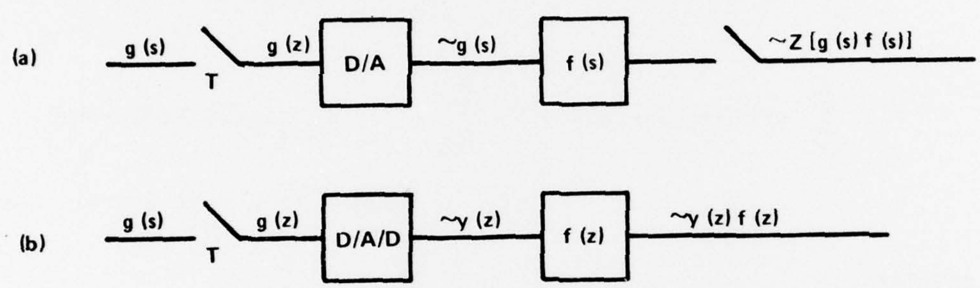

Figure 2. Analog representation of the input.

In this section approximations for constructing a digital representation of an analog input were developed [Figure 2(b)]. The difference is one of point of view.

IV. ANALYSIS OF TUNING

For a given plant, f(s), and input, g(s), the relationship given in Equation (3) is the exact digital representation of the analog input. The approximation,

$$y(z) \simeq T\left[1 - \frac{1}{2}\frac{f_o}{f(z)}\right] g(z) - \frac{T}{2} g_o \quad , \tag{42}$$

would be used to develop recurrence relations for digital simulation (Section V contains the details). For a given plant, f(s), and input, g(s), the ratio of Equations (42) and (3) would be a measure of the accuracy of the approximation.

For an integrator,

$$f(s) = \frac{1}{s} \quad , \tag{43}$$

$$y(z) \simeq \frac{T}{2}(1 + z)\, g(z) - \frac{T}{2} g_o \quad . \tag{44}$$

For the input,

$$g(t) = a\,e^{-at} \tag{45}$$

$$g(s) = \frac{a}{s + a} \quad , \tag{46}$$

and

$$g(z) = \frac{a}{1 - z\,e^{-at}} \quad . \tag{47}$$

The exact representation would be

$$y(z) = \frac{(1 - e^{-at})z}{1 - z\,e^{-at}} \quad . \tag{48}$$

Equation (42) yields

$$y(z) \simeq \frac{T}{2}\,(1 + z)\left(\frac{a}{1 - z\,e^{-at}}\right) - \frac{aT}{2} \tag{49}$$

$$\simeq \frac{\frac{aT}{2}\,(1 + e^{-aT})z}{1 - z\,e^{-aT}} \quad . \tag{50}$$

Dividing Equation (50) by Equation (48) to obtain the ratio of the approximation to the exact output, yields

$$- \frac{aT}{2}\left(\frac{e^{-aT} + 1}{e^{-aT} - 1}\right) \tag{51}$$

which is just

$$\frac{aT}{2}\,\mathrm{ctn}\,\frac{aT}{2} \quad . \tag{52}$$

For small angles

$$\mathrm{ctnh}\,\frac{aT}{2} \simeq \frac{2}{aT} + \frac{aT}{6} + \ \ldots \tag{53}$$

and the ratio would be

$$1 + \frac{(aT)^2}{12} + \ldots \tag{54}$$

of course "a" need not be real; consider

$$a = i\omega \quad , \tag{55}$$

then Equation (52) becomes

$$\frac{\omega T}{2} \operatorname{ctn} \frac{\omega T}{2} \quad , \tag{56}$$

the ratio for a sine wave input to a trapezoidal integrator. For small angles

$$\operatorname{ctn} \frac{\omega T}{2} \simeq \frac{2}{\omega T} - \frac{\omega T}{6} - \ldots \tag{57}$$

and the ratio, Equation (56), is

$$1 - \frac{(\omega T)^2}{12} - \ldots \quad . \tag{58}$$

The ratio, Equation (56), agrees with that obtained by the transfer function approach [9], which also gives the ratio for a sine wave input into some other integrators. Of particular interest is Simpson's rule,

$$\frac{T}{3} \frac{1 + 4z + z^2}{(1 + z)(1 - z)} \tag{59}$$

for which the ratio is

$$\frac{\omega T}{3} \left(\frac{2 + \cos \omega T}{\sin \omega T} \right) \tag{60}$$

and for small angles, Equation (60) becomes

$$1 + \frac{(\omega T)^4}{30} \ldots \tag{61}$$

It is noted that for too large a time step, T, the ratio for an exponential input, Equation (52) would be too large and for a sine wave input, Equation (56) would be too small, This can be corrected by tunable integration. Using tunable convolution, Equation (40),

$$y(z) \simeq \frac{T[\eta(1 - e^{-aT}) - e^{-aT}]z}{1 - z\,e^{-aT}} \tag{62}$$

and the ratio of Equations (62) and (48) is

$$aT\left[\frac{e^{-aT}}{e^{-aT} - 1} - \eta\right] \quad . \tag{63}$$

For "a" real, Equation (63) becomes

$$aT\left[\frac{\cosh\frac{aT}{2} + \sinh\frac{aT}{2}}{2\sinh\frac{aT}{2}} - \eta\right] \tag{64}$$

which simplifies to

$$\frac{aT}{2}\left[\operatorname{ctnh}\frac{aT}{2} + 2\left(\frac{1}{2} - \eta\right)\right] \quad . \tag{65}$$

Solving Equation (65) for η, when the ratio is one,

$$\eta = \frac{1}{2} + \frac{1}{2}\left(\operatorname{ctn}\frac{aT}{2} - \frac{2}{aT}\right) \quad . \tag{66}$$

For $aT < 1$, Equation (64) becomes approximately

$$1 + aT\left(\frac{1}{2} - \eta\right) + \frac{(aT)^2}{12} + \ldots \tag{67}$$

and Equation (66) becomes

$$\eta \simeq \frac{1}{2} + \frac{aT}{12} \quad . \tag{68}$$

For $aT \gg 1$,

$$\eta \simeq 1 - \frac{2}{aT} \quad . \tag{69}$$

For "a" imaginary, Equation (63) becomes

$$\left(\frac{\omega T}{2}\right)\left[\text{ctn}\,\frac{\omega T}{2} + 2i\,\left(\frac{1}{2} - \eta\right)\right] \quad . \tag{70}$$

Because Equation (70) is complex, multiply by the complex conjugate and take the square root, which yields

$$\left(\frac{\omega T}{2}\right)\left[\text{ctn}^2\,\frac{\omega T}{2} + 4\,\left(\frac{1}{2} - \eta\right)^2\right]^{\frac{1}{2}} \tag{71}$$

for the amplitude ratio, and dividing the imaginary part by the real, yields

$$\tan^{-1}\left[2\left(\frac{1}{2} - \eta\right)\tan\frac{\omega T}{2}\right] \quad , \tag{72}$$

the phase error for a sine wave input to a tunable integrator.

For zero phase error,

$$\eta = \frac{1}{2} \quad , \tag{73}$$

the trapezoidal-integrator, and Equation (71) reduces to Equation (56).

For zero amplitude error, the ratio should be one; therefore, setting Equation (71) equal to one

$$1 = \frac{\omega T}{2}\left[\text{ctn}^2\,\frac{\omega T}{2} + 4\,\left(\frac{1}{2} - \eta\right)^2\right]^{\frac{1}{2}} \tag{74}$$

and solving for η,

$$\eta = \frac{1}{2} \pm \frac{1}{2}\left[\left(\frac{2}{\omega T}\right)^2 - \text{ctn}^2\,\frac{\omega T}{2}\right]^{\frac{1}{2}} \quad . \tag{75}$$

For small angles

$$\eta \simeq \frac{1}{2} \pm \frac{1}{2}\left[\left(\frac{2}{\omega T}\right)^2 - \left(\frac{2}{\omega T} - \frac{\omega T}{6}\,\ldots\right)^2\right]^{\frac{1}{2}} \tag{76}$$

$$\simeq \frac{1}{2} \pm \left[\frac{1}{6} - \frac{(\omega T)^2}{144}\right]^{\frac{1}{2}} \quad . \tag{77}$$

and for $\omega T < 1$.

$$\eta \simeq \frac{1}{2} \pm \left(\frac{1}{6}\right)^{\frac{1}{2}} \tag{78}$$

that is,

$$\eta \simeq \begin{cases} 0.9082482905 \\ 0.0917517095 \end{cases} \quad . \tag{79}$$

To determine the ratio and phase error which occurs from ignoring the dependence of η on ωT, Equation (78) is substituted into Equations (71) and (72), and the following are obtained

$$\left(\frac{\omega T}{2}\right)\left[\frac{2}{3} + \operatorname{ctn}^2 \frac{\omega T}{2}\right]^{\frac{1}{2}} \tag{80}$$

and

$$\tan^{-1}\left[2\left(\frac{1}{6}\right)^{\frac{1}{2}} \tan\frac{\omega T}{2}\right] \quad , \tag{81}$$

for the amplitude ratio and phase error, respectively, the ratio for a small angle approximation would be

$$1 + \frac{11}{360}(\omega T)^4 \tag{82}$$

which compares very closely with Simpson's rule, Equation (61), but the phase error would be

$$\left(\frac{1}{6}\right)^{\frac{1}{2}} \omega T \quad .$$

Because $\omega = \frac{2\pi}{P}$

where "p" is the period, it follows that

$$\frac{P}{T} = \frac{2\pi}{\omega T}$$

is the number of samples per period. At the Shannon sampling limit,

$$\frac{P}{T} = 2 \quad ;$$

therefore,

$$\omega T = \pi \quad .$$

The amplitude ratio, Equation (71), is then

$$\pi \left| \frac{1}{2} - \eta \right|$$

at the Shannon limit. For the trapezoidal integrator the ratio would be zero; the cost of zero phase error.

Simpson's rule also has no phase error but at the Shannon limit the ratio, Equation (60), would be infinity.

For Equation (78), Equation (80) gives the ratio 1.28 ..., a 28% amplitude error for small angle tuning at the Shannon limit better than Simpon's rule.

For unity amplitude ratio at the Shannon limit,

$$\eta \simeq \frac{1}{2} \pm \frac{1}{\pi} \tag{83}$$

that is,

$$\eta = \begin{cases} 0.8183098861 \\ 0.1816901139 \end{cases} \quad . \tag{84}$$

For Equation (83), Equations (71) and (72) become respectively,

$$\left(\frac{\omega T}{2}\right) \left[\operatorname{ctn}^2 \frac{\omega T}{2} + \frac{4}{\pi^2} \right]^{\frac{1}{2}} \quad , \tag{85}$$

and

$$\tan^{-1} \left[\pm \frac{2}{\pi} \tan \frac{\omega T}{2} \right] \quad . \tag{86}$$

For small angles, the ratio, Equation (86), would be approximately

$$1 - \frac{(\omega T)^2}{15.3} \quad , \tag{87}$$

not much better than trapezoidal, Equation (58).

Thus far, only exponential inputs to an integrator have been considered. Since functions may be expressed in Taylor series,

$$f(t) = \sum_{n=0}^{\infty} fo^{(n)} \frac{t^n}{n!} \quad , \tag{88}$$

another input of interest would be powers of t.

For

$$g(t) = 1 \quad , \tag{89}$$

$$g(s) = \frac{1}{s} \tag{90}$$

and trapezoidal convolution yields

$$Z\left(\frac{1}{s} \cdot \frac{1}{s}\right) = \frac{T\ z}{(1 - z)^2} \tag{91}$$

which is exact and the ratio is one.

For

$$g(t) = t \quad ,$$

$$g(s) = \frac{1}{s^2} \tag{93}$$

and trapezoidal convolution yields

$$Z\left(\frac{1}{s^2} \cdot \frac{1}{s}\right) = \frac{T^2 z(1 + z)}{2(1 - z)^3} \tag{94}$$

which is also exact and again the ratio is one. This in not surprising because a trapezoidal integrator can integrate a constant or a ramp perfectly for any size time step.

For

$$g(t) = t^2 \quad , \tag{95}$$

$$g(s) = \frac{2!}{s^3} \tag{96}$$

and

$$Z\left(\frac{2!}{s^4}\right) = \frac{T^3\ z(1 + 4z + z^2)}{3(1 - z)^4} \tag{97}$$

and, trapezoidal convolution yields

$$Z\left(\frac{2!}{s^3} \cdot \frac{1}{S}\right) \simeq \frac{T^3\ z(1 + 2z + z^2)}{2(1 - z)^4} \tag{98}$$

Dividing Equation (98) by Equation (97) yields

$$\frac{3}{2} \cdot \frac{1 + 2z + z^2}{1 + 4z + z^2} \quad . \tag{99}$$

The "z"s do not cancel out as in the previous examples. Applying the final value theorem, in the limit as z approaches one, Equation (99) approaches one; but applying the initial value theorem, as z approaches zero, Equation (99) approaches three halves, a 50% error.

Substituting the definition of the z-transform into

$$Z\left(\frac{g(s)}{s}\right) \simeq \frac{T[\eta + (1 - \eta)z]}{1 - z}\ g(z) \tag{100}$$

the following is obtained:

$$(1 - z) \sum_{n=0}^{\infty} \mathcal{L}^{-1}\left(\frac{g(s)}{s}\right)\Bigg/_{t\,=\,nT} z^n \simeq T[\eta + (1 - \eta)z] \sum_{n=0}^{\infty} g(nT)z^n \tag{101}$$

which may be reduced to

$$\sum_{n=0}^{\infty} \mathcal{L}^{-1}\left(\frac{g(s)}{s}\right)\Bigg/_{t\,=\,nT} z^n - \sum_{n=1}^{\infty} \mathcal{L}^{-1}\left(\frac{g(s)}{s}\right)\Bigg/_{t\,=\,(n\,-\,1)T} z^n$$

$$\simeq \eta T \sum_{n=0}^{\infty} g(nT)z^n - (1 - \eta)T \sum_{n=0}^{\infty} g(nT - T)z^n \quad . \tag{102}$$

For

$$g(t) = \frac{t^2}{2} \quad , \tag{103}$$

$$y(s) = \frac{1}{s^3} \tag{104}$$

and

$$\mathcal{L}^{-1}\left(\frac{g(s)}{s}\right) = \frac{t^3}{6} \quad ; \tag{105}$$

Equation (102) becomes, after equating coefficients of like powers of z,

$$\frac{1}{6}\left[(nT)^3 - (nT - T)^3\right] \simeq \frac{T}{2}\left[\eta(nT)^2 + (1 - \eta)(nT - T)^2\right] \quad , n > 0 \quad . \tag{106}$$

taking the difference between the right side and the left side to determine the error per time step, the following is obtained:

$$T^3\left[\left(\frac{1}{2} - \eta\right)n + \left(\frac{1}{6} - \frac{1 - \eta}{2}\right)\right] \tag{107}$$

which implies, for zero error, that

$$\eta = \frac{1}{2}\left(\frac{n - \frac{2}{3}}{n - \frac{1}{2}}\right) \quad . \tag{108}$$

For $n = 1$, $\eta = \frac{1}{3}$ and it is clear why the initial value for the trapezoidal integrator, $\eta = \frac{1}{2}$, is three halves. It is noted that η is different for each step.

Summing the error per step for m steps

$$T^3 \left[\left(\frac{1}{2} - \eta\right) \sum_{n=1}^{m} n + \left(\frac{3\eta - 2}{6}\right) \sum_{n=1}^{m} 1 \right] \tag{109}$$

gives the total error for m steps,

$$T^3 \left[\left(\frac{1}{2} - \eta\right) \frac{m}{2} (m + 1) + \left(\frac{\eta}{2} - \frac{1}{3}\right) m \right] \quad , \; m > 0 \quad . \tag{110}$$

For the total error for m steps to be zero,

$$\eta = \frac{1}{2} \left(\frac{m - \frac{1}{3}}{m}\right) \tag{111}$$

$$n = \frac{1}{2} (m + 1) \quad ,$$

Equation (111) is in agreement with Equation (108). Therefore, the η which tunes for m steps is exact for the middle step, and the errors for each step are averaged out. The mean value theorem applies for one step, Equation (108), or m steps taken together, Equation (111). Because it is not practical to adjust the timing for each step, an overall tuning using Equation (111) is indicated. This leaves the initial value of the ratio uncorrected.

In the earlier examples where the z's cancelled out when the ratio was taken, it would appear that η is the same for all steps, because there was no dependence on the step number. That this occurs for some functions seems to be the power behind tunable integration.

The analysis of an exponential input to an integrator would also apply to a constant into a single pole filter because the product of the Laplace transforms is the same. A damped sine wave and a single pole filter, etc. could also be analyzed.

V. SOME SAMPLE PROBLEMS

The following relationships from Laplace transforms will be found helpful in developing recurrence relations:

$$\mathcal{L}\left[x^{(n)}(t)\right] = s^{n}x(s) - \sum_{\ell=0}^{n-1} s^{n-\ell-1}\, x_o^{(\ell)} \tag{112}$$

and

$$\mathcal{L}\left[x^{(-n)}(t)\right] = \frac{x(s)}{s^n} + \sum_{\ell=1}^{n} \frac{x_o^{(-\ell)}}{s^{n-\ell+1}} \tag{113}$$

When the initial conditions are nonzero, one should proceed from the differential equation using these relationships, and not directly from the transfer function.

The z-transform of

$$Z\left[x_o^{(\ell)}\, g(s)\right] = \sum_{n=0}^{\infty} z^n \int_o^{nT} x_o^{(\ell)}\, \delta(t)\, g(nT - t)\, dt \tag{114}$$

because a constant in the frequency domain transforms to a Dirac delta in the time domain. From the properties of the Dirac delta (Appendix C) Equation (114) becomes

$$Z\left[x_o^{(\ell)}\, g(s)\right] = \sum_{n=0}^{\infty} z^n\, x_o^{(\ell)}\, g(nT) \tag{115}$$

$$= x_o^{(\ell)}\, g(z) \quad . \tag{116}$$

This relationship, Equation (116), will be used in incorporating initial conditions into the recurrences.

A. Single Integration

From Equation (113) for n = 1,

$$x(s) = \frac{x(s) + x_o}{s} \quad . \tag{117}$$

Taking the z-transform of Equation (117), the following is obtained:

$$x(z) = Z\left(\frac{\dot{x}(s)}{s}\right) + x_o\, Z\left(\frac{1}{s}\right) \quad . \tag{118}$$

Using trapezoidal convolution, Equation (30),

$$Z\left(\frac{\dot{x}(s)}{s}\right) \simeq \frac{T}{2}\left(\frac{1+z}{1-z}\right)\dot{x}(z) - \frac{T}{2}\left(\frac{\dot{x}_o}{1-z}\right) \tag{119}$$

From the definition of the z-transform

$$(1-z)\sum_{n=0}^{\infty} x(nT)z^n \simeq \frac{T}{2}(1+z)\sum_{n=0}^{\infty}\dot{x}(nT)z^n - \frac{T}{2}\dot{x}_o + x_o \quad . \tag{120}$$

$$\sum_{n=0}^{\infty}[x(nT)z^n - x(nT)z^{n+1}] \simeq \frac{T}{2}\sum_{n=0}^{\infty}[\dot{x}(nT)z^n + \dot{x}(nT)z^{n+1}] - \frac{T}{2}\dot{x}_o + x_o \quad . \tag{121}$$

$$\sum_{n=0}^{\infty} x(nT)z^n - \sum_{n=1}^{\infty} x(nT-T)z^n \simeq \frac{T}{2}\sum_{n=0}^{\infty}\dot{x}(nT)z^n + \frac{T}{2}\sum_{n=1}^{\infty}\dot{x}(nT-T)z^n - \frac{T}{2}\dot{x}_o + x_o \quad . \tag{122}$$

Equating coefficients of powers of z,

$$x_o = x_o \qquad n = 0 \tag{123}$$

$$x_n = x_{n-1} + \frac{T}{2}(\dot{x}_n + \dot{x}_{n-1}),\ n > 0 \quad , \tag{124}$$

the trapezoidal integrator.

In a similar fashion for tunable convolution, Equation (40),

$$Z\left(\frac{\dot{x}(s)}{s}\right) \simeq \frac{T[\eta + (1-\eta)z]}{1-z}\dot{x}(z) - T\eta\left(\frac{\dot{x}_o}{1-z}\right) \tag{125}$$

and Equation (125) becomes

$$(1-z)x(z) \simeq T[\eta + (1-\eta)z]\dot{x}(z) - T\eta x_o + x_o \quad . \tag{126}$$

It follows that, after manipulations similar to those in Equations (120) to (122),

$$X_o = X_o \qquad , \; n = 0 \quad ,$$

$$X_n = x_{n-1} + T\,[\eta\,\dot{x}_n + (1 - \eta)\dot{x}_{n-1}],\; n > 0 \quad , \tag{128}$$

the tunable integrator.

To illustrate the difference between a constant in the time and frequency domain, the following differential equation is considered

$$\dot{x}(t) = k \quad . \tag{129}$$

In the frequency domain, Equation (129) becomes

$$s\,x(s) - x_o = \frac{k}{s} \tag{130}$$

and solving for x(s),

$$x(s) = \frac{\frac{k}{s} + x_o}{s} \quad . \tag{131}$$

Transforming Equation (131) back to the time domain, the following is obtained:

$$x(t) = \int_o^t \left[k + x_o\;\delta(\tau)\right] d\tau \tag{132}$$

$$= k \int_o^t d\tau + x_o \tag{133}$$

$$= kt + x_o \quad . \tag{134}$$

Instead, the z-transform is taken of

$$x(s) = \frac{k}{s^2} + \frac{x_o}{s} \quad , \tag{135}$$

which would be

$$x(z) = \frac{kTz}{(1 - z)^2} + \frac{x_o}{1 - z} \tag{136}$$

$$= \frac{kTz + (1 - z)\, x_o}{(1 - z)^2} \tag{137}$$

The recurrence would be

$$x_o = x_o \tag{138}$$

$$x_1 = x_o + kT \tag{139}$$

$$x_n = 2\, x_{n-1} - x_{n-2},\ n > 1 \quad . \tag{140}$$

This recurrence may be unexpected but it should be tried because it works. Consider

$$x(z) = \frac{1}{1 - z} \left[\frac{kTz}{1 - z} + x_o\right] \tag{141}$$

and noting that

$$\frac{1}{1 - z} = \sum_{n=0}^{\infty} z^n \tag{142}$$

one may also have the recurrence

$$x_o = x_o \qquad , n = 0 \quad , \tag{143}$$

$$x_n = x_{n-1} + kT,\ n > 0 \quad . \tag{144}$$

This recurrence has an advantage in that no difference is taken and only one step is required to start, though both are proper, that is, correct recurrences.

Though not necessary because k is a known function and the exact z-transform may be taken, it is instructive to apply trapezoidal convolution to Equation (131).

$$Z\left(\frac{1}{s} \cdot \frac{k}{s}\right) = TZ\left(\frac{1}{s}\right) Z\left(\frac{k}{s}\right) - \frac{T}{2}\left[kZ\left(\frac{1}{s}\right) + 1 \cdot Z\left(\frac{k}{s}\right)\right] \tag{145}$$

$$= T\left(\frac{1}{1-z}\right)\left(\frac{k}{1-z}\right) - \frac{T}{2}\left[k\left(\frac{1}{1+z}\right)+\left(\frac{k}{1-z}\right)\right] \tag{146}$$

$$= \frac{T}{2}\left(\frac{1+z}{1-z}\right)\left(\frac{k}{1-z}\right) - \frac{T}{2}\,k\left(\frac{1}{1-z}\right) \tag{147}$$

Because k is a constant in the time domain, its initial condition and its value at any time are the same. For instructive purposes, they will be treated as being different.

Noting that

$$\frac{k}{1-z} = k\sum_{n=0}^{\infty} z^n \equiv k(z) \tag{148}$$

and Equation (147) becomes,

$$x(z) = \frac{T}{2}\left(\frac{1+z}{1-z}\right) k(z) - \frac{T}{2}\,k\left(\frac{1}{1-z}\right) + \frac{x_o}{1-z} \quad . \tag{149}$$

the recurrence would be

$$x_o = x_o \tag{150}$$

$$x_n = x_{n-1} + \frac{T}{2}(k + k) \tag{151}$$

$$= x_{n-1} + Tk \quad . \tag{152}$$

the initial value of x(t) is x_o, but it only takes on that value at $t = 0$ $(n = 0)$, unless $k = 0$, in which case

$$x_o = x_o \quad , n = 0 \quad , \tag{153}$$

$$x_n = x_{n-1}, \ n > 0 \quad . \tag{154}$$

B. Double Integration

From Equation (113) for n = 2,

$$x(s) = \frac{\ddot{x}(s)}{s^2} + \frac{\dot{x}_o}{s^2} + \frac{x_o}{s} \tag{155}$$

and taking the z-transform

$$x(z) = Z\left(\frac{\ddot{x}(s)}{s^2}\right) + \dot{x}_o\, Z\left(\frac{1}{s^2}\right) + x_o\, Z\left(\frac{1}{s}\right) \tag{156}$$

Using trapezoidal convolution, Equation (30),

$$Z\left(\frac{\ddot{x}(s)}{s^2}\right) \simeq T\left(\frac{Tz}{(1 - z)^2}\right) \ddot{x}(z) - \frac{T}{2}\,\frac{Tz\ddot{x}_o}{(1 - z)^2} \tag{157}$$

and Equation (156) becomes

$$(1 - z)^2\, x(z) \simeq T^2 z\ddot{x}(z) - \frac{T^2}{2}\, z\ddot{x}_o + Tz\dot{x}_o + (1 - z)x_o \quad . \tag{158}$$

Equating coefficients of powers of z,

$$x_o = x_o \qquad , n = 0 \quad , \tag{159}$$

$$x_1 = x_o + T\,\dot{x}_o + \frac{T^2}{2}\,\ddot{x}_o, \; n = 1 \quad , \tag{160}$$

and

$$x_n = 2x_{n-1} - x_{n-2} + T^2\,\ddot{x}_{n-1}, \; n > 1 \quad . \tag{161}$$

In general, the number of steps before the recurrence for x_n may be applied is equal to the order of the denominator of the "transfer function." The required information for the start-up steps is contained in the initial conditions, even when they are zero (Appendix D).

The recurrence, Equation (161), does not appear desirable numerically because a difference is required and it takes two steps to start. Because no feedback is involved, two single integrators could be used to overcome these difficulties;

$$\dot{x}_o = \dot{x}_o \qquad , n = 0 \tag{162}$$

$$\dot{x}_n = \dot{x}_{n-1} + \frac{T}{2}(\ddot{x}_n + \ddot{x}_{n-1}), \; n > 0 \tag{163}$$

and

$$x_o = x_o \qquad , n = 0 \tag{164}$$

$$x_n = x_{n-1} + \frac{T}{2}(\dot{x}_n + \dot{x}_{n-1}), \; n > 0 \quad . \tag{165}$$

C. A Single Pole Filter

Consider the following differential equation

$$\dot{x}(t) = -a\, x(t) + g(t) \quad . \tag{166}$$

This would transform to

$$s\, x(s) - x_o = -a\, x(s) + g(s) \tag{167}$$

and finally

$$x(s) = \frac{g(s) + x_o}{s + a} \tag{168}$$

in the frequency domain. Taking the z-transform of Equation (168),

$$x(z) = Z\left(\frac{g(s)}{s + a}\right) + x_o\, Z\left(\frac{1}{s + a}\right) \tag{169}$$

Using trapezoidal convolution, Equation (30),

$$Z\left(\frac{g(s)}{s + a}\right) \simeq \frac{T}{2}\left(\frac{1 + ze^{-aT}}{1 - ze^{-aT}}\right) g(z) - \frac{T}{2}\left(\frac{g_o}{1 - ze^{-aT}}\right) \tag{170}$$

and Equation (169) becomes

$$(1 - ze^{-aT})x(z) \simeq \frac{T}{2}(1 + ze^{-aT})g(z) - \frac{T}{2}g_o + x_o \quad . \tag{171}$$

Equating coefficients of powers of z,

$$x_o = x_o \qquad , n = 0 \tag{172}$$

$$x_n \simeq e^{-aT} x_{n-1} + \frac{T}{2}\left[g_n + e^{-aT} g_{n-1}\right], \; n > 0 \quad . \tag{173}$$

The z-transform of the following equation might have been used instead

$$x(s) = \frac{g(s) - a\,x(s) + x_o}{s} \tag{174}$$

which is

$$x(z) = Z\left(\frac{g(s) - ax(s)}{s}\right) + x_o\, Z\left(\frac{1}{s}\right) \quad . \tag{175}$$

The recurrence would then be

$$x_o = x_o \qquad , n = 0 \tag{176}$$

$$x_n \simeq x_{n-1} + \frac{T}{2}\left[g_n - ax_n + g_{n-1} - a\,x_{n-1}\right], \; n > 0 \tag{177}$$

and solving for x_n,

$$x_n \simeq \left(\frac{1 - \frac{aT}{2}}{1 + \frac{aT}{2}}\right) x_{n-1} + \frac{T}{2}\left(\frac{g_n + g_{n-1}}{1 + \frac{aT}{2}}\right), \; n > 0 \tag{178}$$

It is noted that [10]

$$\left(\frac{1 - \frac{aT}{2}}{1 + \frac{aT}{2}}\right) = (1/1) \text{ Padé approximation for } e^{-aT} \tag{179}$$

and

$$\left(\frac{1}{1 + \frac{aT}{2}}\right) = (0/1) \text{ Padé approximation for } e^{-\frac{aT}{2}} \quad . \tag{180}$$

Instead of solving for x_n, a past value, x_{n-1}, might be used, that is,

$$x_n \simeq x_{n-1} + \frac{T}{2}\left[g_n - a\,x_{n-1} + g_{n-1} - a\,x_{n-1}\right] \tag{181}$$

which would lead to

$$x_n \simeq (1 - aT)\,x_{n-1} + \frac{T}{2}\left[g_n + g_{n-1}\right] \tag{182}$$

and it is noted that [10]

$$(1 - aT) = (1/0) \text{ Padé approximation for } e^{-aT} \quad . \tag{183}$$

and

$$1 = (0/0) \text{ Padé approximation for } e^{-aT} \quad . \tag{184}$$

The declining accuracy of the approximation is also noted.

Additionally, Equation (182) could be used to predict x_n, substitute Equation (182) into the right side of Equation (177),

$$x_n \simeq x_{n-1} + \frac{T}{2}\left[g_n + g_{n-1}\right] - \frac{aT}{2}\left[(1 - aT)x_{n-1} + \frac{T}{2}(g_n + g_{n-1}) + x_{n-1}\right] \tag{185}$$

and simplyfing

$$x_n \simeq \left(1 - aT + \frac{(aT)^2}{2}\right) x_{n-1} + \frac{T}{2}\left(1 - \frac{aT}{2}\right)\left[g_n + g_{n-1}\right] \tag{186}$$

where

$$\left(1 - aT + \frac{(aT)^2}{2}\right) = (2/0) \text{ Padé for } e^{-aT} \tag{187}$$

and

$$\left(1 - \frac{aT}{2}\right) = (1/0) \text{ Padé for } e^{-\frac{aT}{2}} \quad . \tag{188}$$

In these cases where "a" is time varying, and it is not desired to compute e^{-aT} at each time step, the desired Pade´ approximation should be substituted directly into Equation (173).

D. A Forced Damped Oscillator

The differential equation is

$$\ddot{x}(t) + 2\,\zeta\,\omega_o\,\dot{x}(t) + \omega_o^{\,2}\,x(t) = g(t) \tag{189}$$

where ω_o is the undamped natural frequency and ζ is the damping factor. From Equation (112)

$$\dot{x}(s) = s\,x(s) - x_o \tag{190}$$

and

$$\ddot{x}(s) = s^2\,x(s) - s\,x_o - \dot{x}_o \quad . \tag{191}$$

Substituting Equations (190) and (191) into Equation (189) and solving for x(s),

$$x(s) = \frac{g(s) + (s + 2\,\zeta\,\omega_o)x_o + \dot{x}_o}{s^2 + 2\,\zeta\,\omega_o\,s + \omega_o^{\,2}} \quad . \tag{192}$$

It is noted that

$$\frac{x(s)}{g(s)} = \frac{1}{s^2 + 2\,\zeta\,\omega_o\,s + \omega_o^{\,2}} \tag{193}$$

only if $x_o = 0$ and $\dot{x}_o = 0$.

Three possible ways of implementing Equations (192) and (193) are shown in Figure 3. Figure 3(c) would be a proper implementation on an analog computer and is usually the approach taken in digital simulation.

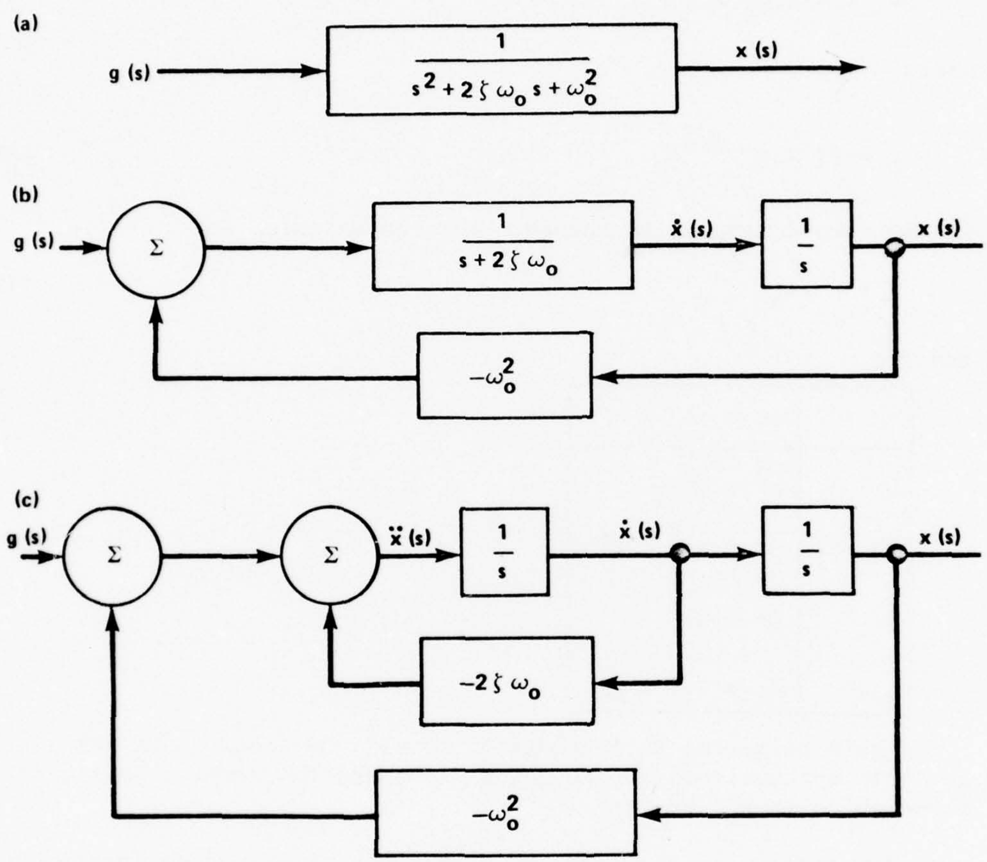

Figure 3. Implementation of Equations (192) and (193).

A perusal of Appendix B will reveal no z-transform for Equation (193) but

$$s^2 + 2\,\zeta\,\omega_o\,s + \omega_o^{\,2} = (s + \zeta\,\omega_o)^2 + \omega^2 \tag{194}$$

where

$$\omega = (1 - \zeta)^{1/2}\,\omega_o \tag{195}$$

is the damped natural frequency. For convenience, let

$$a = \zeta\,\omega_o \tag{196}$$

and for

ζ	$(s + a)^2 + \omega^2$
0	$s^2 + \omega^2$
(0, 1)	$(s + a)^2 + \omega^2$
1	$(s + a)^2$
1 <	$(s + a)^2 - \omega^2$

(197)

Again referring to Appendix B, it will be noted there are four possible z-transforms for Equation (193) and the damping factor determines which should be used.

The case where there is no damping, $\zeta = 0$, and no forcing function, $g(t) = 0$, may be of interest if a free running oscillator is required. In this case, Equation (192) reduces to

$$x(s) = \frac{s x_o + \dot{x}_o}{s^2 + \omega^2} \tag{198}$$

and taking the z-transform

$$x(z) = x_o\, Z\left(\frac{s}{s^2 + \omega^2}\right) + \dot{x}_o\, Z\left(\frac{1}{s^2 + \omega^2}\right) \tag{199}$$

which becomes

$$x(z) = \frac{x_o\ (1 - z\cos\omega T) + \dot{x}_o\ (z \sin\omega T)/\omega}{1 - (2\cos\omega T)z + z^2} \quad . \tag{200}$$

Substituting the definition of the z-transform for x(z), Equation (200) becomes

$$\sum_{n=0}^{\infty} x(nT)z^n - 2\cos\omega T \sum_{n=1}^{\infty} x(nT - T)z^n + \sum_{n=2}^{\infty} x(nT - 2T)z^n$$
$$= x_o + [-(\cos\omega T)x_o + \dot{x}_o(\sin\omega T)/\omega]z \tag{201}$$

which may be written

$$\sum_{n=0}^{\infty} \left[x(nT) - (2\cos\omega T)x(nT - T) + x(nT - 2T)\right]z^n$$
$$+ (2\cos\omega T)x(-T) - x(-2T) - x(-T)z$$
$$= x_o + [-(\cos\omega T)x_o + \dot{x}_o(\sin\omega T)/\omega]z \tag{202}$$

Equating coefficients of power of z,

$$x_o = x_o \qquad , n = 0 \tag{203}$$

$$x_1 = x_o\ (\cos\omega T) + \dot{x}_o\ (\sin\omega T)/\omega,\ n = 1 \tag{204}$$

$$x_n = (2\cos\omega T)x_{n-1} - x_{n-2} \qquad , n > 1 \quad . \tag{205}$$

The state transition method yields the following recurrences [5],

$$\dot{x}_n = (\cos\omega T)\dot{x}_{n-1} - (\omega \sin\omega T)x_{n-1} \quad , \tag{206}$$

and

$$x_n = \left(\frac{\sin\omega T}{\omega}\right)\dot{x}_{n-1} + (\cos\omega T)x_{n-1} \quad . \tag{207}$$

In comparing Equation (205) with Equations (206) and (207), it is noted that the state transition method requires four multiplications, one addition, and one subtraction and two stores, while Equation (205) requires one multiplication, one subtraction, and two stores after a one step start up, Equation (204). The start up step, Equation (204), is the same as Equation (207).

These recurrences are exact because there was no requirement to invoke any approximations. The exactness is easily demonstrated. If

$$x = \cos(\omega T + \phi) \tag{208}$$

it follows that

$$\dot{x} = -\omega \sin(\omega t + \phi) \tag{209}$$

and for $t = nT$,

$$x_o = \cos \phi \qquad , n = 0 \quad , \tag{210}$$

$$x_1 = \cos \omega T \cos \phi - \sin \omega T \sin \phi \quad ,$$

$$= \cos (\omega T + \phi) \qquad , n = 1 \quad , \tag{211}$$

and

$$x_{n+1} = 2 \cos \omega T \cos(n\omega T + \phi) - \cos((n - 1)\omega T + \phi) = \cos ((n + 1)\omega T + \phi), \quad n \geq 1 \quad . \tag{212}$$

Of the four possibilities, Equation (197), the case where the damping factor, ζ, is on the interval between zero and one is the most useful.

$$\begin{aligned} x(z) \simeq {} & T\, Z\left(\frac{1}{(s + a)^2 + \omega^2}\right)\left(g(z) - \frac{1}{2} g_o\right) \\ & + x_o\, Z\left(\frac{s + 2a}{(s + a)^2 + \omega^2}\right) \\ & + \dot{x}_o\, Z\left(\frac{1}{(s + a)^2 + \omega^2}\right) \end{aligned} \tag{213}$$

taking the z-transforms, the following is obtained:

$$
\begin{aligned}
&(1 - 2z\, e^{-aT} \cos \omega T + z^2\, e^{-2aT}) x(z) \\
&\quad = T\, z e^{-aT} \frac{\sin \omega T}{\omega} \left[g(z) - \frac{1}{2} g_o \right] \\
&\quad + x_o \left(1 - z\, e^{-aT} \cos \omega T \quad az\, e^{-aT} \frac{\sin \omega T}{\omega} \right) \\
&\quad + \dot{x}_o \left(z\, e^{-aT} \frac{\sin \omega T}{\omega} \right) \quad .
\end{aligned} \tag{214}
$$

Equating coefficients of z in Equation (214) after substituting the definition of the z-transform,

$$x_o = x_o \qquad , \; n = 0 \quad , \tag{215}$$

$$
\begin{aligned}
x_1 &= \frac{T}{2} e^{-aT} \frac{\sin \omega T}{\omega} g_o \\
&+ e^{-aT} \left(\cos \omega T + \frac{a}{\omega} \sin \omega T \right) x_o \\
&+ e^{-aT} \frac{\sin \omega T}{\omega} \dot{x}_o \qquad , \; n = 1 \quad ,
\end{aligned} \tag{216}
$$

$$
\begin{aligned}
x_n &= 2e^{-aT} \cos \omega T\; x_{n-1} - e^{-2aT} x_{n-2} \\
&+ T\, e^{-aT} \frac{\sin \omega T}{\omega} g_{n-1} \qquad , \; n > 1 \quad .
\end{aligned} \tag{217}
$$

The same procedure would be followed in developing recurrences for the other values of the damping factor, ζ, in Equation (197).

One problem, the values of $\dot{x}_n$ and $\ddot{x}_n$ may be desired. This leads to a combination of Figures 3a and 3b (Figure 4).

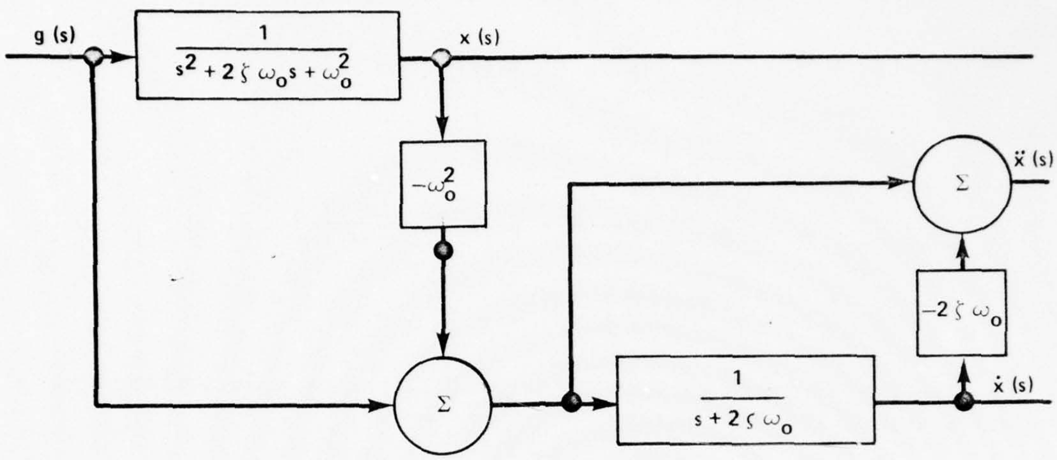

Figure 4. Figures 3(a) and 3(b) combined.

Writing Equation (189) as follows

$$\ddot{x}(t) + 2\zeta\omega_o\dot{x}(t) = g(t) - \omega_o^2 x(t) \tag{218}$$

and using Equation (112), the following is obtained:

$$(s + 2\zeta\omega_o)\dot{x}(s) = g(s) - \omega_o^2 x(s) + \dot{x}_o \quad . \tag{219}$$

and finally

$$\dot{x}(s) = \frac{g(s) - \omega_o^2 x(s) + \dot{x}_o}{s + 2\zeta\omega_o} \quad . \tag{220}$$

This is the single pole filter discussed in the previous section. Taking the z-transform of Equation (220), the following is obtained:

$$\begin{aligned}(1 - z\,e^{-2aT})\dot{x}(z) &\simeq \frac{T}{2}(1 + z\,e^{-2aT})\left[g(z) - \omega_o^2\, x(z)\right] \\ &- \frac{T}{2}\left(g_o - \omega_o^2\, x_o\right) + \dot{x}_o\end{aligned} \tag{221}$$

and the recurrence would be,

$$\dot{x}_o = \dot{x}_o \qquad , \; n = 0 \quad , \tag{222}$$

and

$$\dot{x}_n \simeq e^{-2aT} x_{n-1} + \frac{T}{2} \left[g_n - \omega_o^2 x_n + e^{-2aT} \left(g_{n-1} - \omega_o^2 x_{n-1} \right) \right], \quad n > 0. \tag{223}$$

x_n would be computed using Equations (216) and (217); then $\dot{x}_n$ would be computed using Equation (223) and finally $\ddot{x}_n$ using Equation (224),

$$\ddot{x}_n = g_n - 2\,\zeta\,\omega_o \dot{x}_n - \omega_o^2 x_n \quad . \tag{224}$$

Because the proper recurrence for x_n is a function of the damping factor, ζ, the preceding approach may be too complex. Let

$$x_n = x_{n-1} + \frac{T}{2} (\dot{x}_n + \dot{x}_{n-1}) \tag{225}$$

and substituting Equation (225) into Equation (223), the following is obtained:

$$\begin{aligned} \dot{x}_n = e^{-2aT} \dot{x}_{n-1} + \frac{T}{2} \Bigg\{ & g_n - \omega_o^2 \left[x_{n-1} + \frac{T}{2} (\dot{x}_n + \dot{x}_{n-1}) \right] \\ & + e^{-2aT} (g_{n-1} - \omega_o^2 x_{n-1}) \Bigg\} \end{aligned} \tag{226}$$

and solving for $\dot{x}_n$,

$$\dot{x}_n = \frac{\left[e^{-2aT} - \left(\frac{\omega_o T}{2} \right)^2 \right] \dot{x}_{n-1} + \frac{T}{2} \left[g_n + e^{-2aT} g_{n-1} - \omega_o^2 (1 + e^{-2aT}) x_{n-1} \right]}{1 + \left(\frac{\omega_o T}{2} \right)^2} \quad . \tag{227}$$

This would correspond to Figure 3(b).

Another way to develop a recurrence would be to integrate twice; this is the more traditional approach. Let

$$\dot{x}_n = \dot{x}_{n-1} + \frac{T}{2} (\ddot{x}_n + \ddot{x}_{n-1}) \tag{228}$$

and

$$x_n = x_n + \frac{T}{2} (\dot{x}_n + \dot{x}_{n-1}) \tag{229}$$

where

$$\ddot{x}_n = g_n - 2 \zeta \omega_o \dot{x}_n - \omega_o^2 x_n \quad . \tag{230}$$

Because there are three equations and three unknowns, $\ddot{x}_n$, $\dot{x}_n$, and x_n, Equation (230) is substituted; then Equation (229) is substituted into Equation (228) and solved for $\dot{x}_n$,

$$\dot{x}_n \simeq \left[\frac{1 - \zeta\omega_o T - \left(\frac{\omega_o T}{2}\right)^2}{1 + \zeta\omega_o T + \left(\frac{\omega_o T}{2}\right)^2}\right] \dot{x}_{n-1} + \frac{T}{2}\left[\frac{g_n + g_{n-1} - 2\ \omega_o^2\ x_{n-1}}{1 + \zeta\omega_o T + \left(\frac{\omega_o T}{2}\right)^2}\right] \quad . \tag{231}$$

This would correspond to Figure 3(c).

E. Newtonian Drag

A nonlinear problem of interest in missile and aircraft simulations is the Riccati equation representing Newtonian drag with acceleration, Equation (232):

$$\dot{u}(t) = -k(t)\ u^2(t) + g(t) \quad , \tag{232}$$

which in the frequency domain may be written

$$s\ u(s) - u_o = -\ \mathcal{L}[k(t)u^2(t)] + g(s) \tag{233}$$

$$u(s) = \frac{-\ \mathcal{L}[k(t)u^2(t)] + g(s)}{s} \quad . \tag{234}$$

Taking the z-transform of Equation (234) the following is obtained:

$$u(z) = -Z\left(\frac{\mathcal{L}[k(t)u^2(t)]}{s}\right) + Z\left(\frac{g(s)}{s}\right) + u_o\ Z\left(\frac{1}{s}\right) \quad . \tag{235}$$

With the help of trapezoidal convolution, Equation (30), Equation (235) becomes

$$(1 - z)\ u(z) = \frac{T}{2}\ (1 + z)\ Z\left\{g(s) - \mathcal{L}\left[k(t)u^2(t)\right]\right\} - \frac{T}{2}\left[g_o - k_o\ u_o^{\ 2}\right]$$

$$+\ u_o \quad . \tag{236}$$

Equating coefficients of powers of z, the following is obtained:

$$u_o = u_o \qquad , n = 0 \quad , \tag{237}$$

and

$$u_n = u_{n-1} + \frac{T}{2}\left[g_n + g_{n-1}\right] - \frac{T}{2}\left[k_n\ u_n^{\ 2} + k_{n-1}\ u_{n-1}^2\right] \quad , n > 0 \quad . \tag{238}$$

$$\frac{T}{2}\ k_n u_n^{\ 2} + u_n = \left(1 - \frac{T}{2}\ k_{n-1}\ u_{n-1}\right)\ u_{n-1} + \frac{T}{2}\ \ g_n + g_{n-1} \tag{239}$$

Letting

$$c_n = \left(1 - \frac{T}{2}\ k_{n-1}\ u_{n-1}\right) u_{n-1} + \frac{T}{2}\ \ g_n + g_{n-1} \tag{240}$$

and solving for u_n, the following is obtained:

$$u_n = \frac{2\ c_n}{1 + (1 + 2Tk_n\ c_n)^{1/2}} \quad , n > 0 \quad . \tag{241}$$

This recurrence, Equation (241), seems a bit complex. Letting

$$k_n\ u_n^{\ 2} = k_n\left[u_{n-1} + \left(u_n - u_{n-1}\right)\right]^2 \tag{242}$$

$$= -k_n\ u_{n-1}^2 + 2k_n\ u_{n-1}\ u_n + k_n\left(u_n - u_{n-1}\right)^2 \tag{243}$$

and assuming that $\left(u_n - u_{n-1}\right)^2$ is negligible

$$k_n\ u_n^{\ 2} \simeq -k_n\ u_{n-1}^2 + 2k_n\ u_{n-1}\ u_n \quad . \tag{244}$$

Substituting Equation (244) into Equation (238) and solving for u_n, yields

$$u_n \simeq \frac{\left[1 + \frac{T}{2}\left(k_n - k_{n-1}\right) u_{n-1}\right] u_{n-1} + \frac{T}{2}\left[g_n + g_{n-1}\right]}{1 + Tk_n u_{n-1}} \quad , \; n > 0 \tag{245}$$

(245)

A small angle approximation to the analytic solution of Equation (232), for g and k constant, is [11],

$$u \simeq \frac{u_o + gt}{1 + t\, k\, u_o} \quad . \tag{246}$$

It would appear that Equation (245) is a better approximation than Equation (241) when compared with the analytic solution [11].

Another possibility is to use the relationship between the Riccati equation and the linear differential equation of second order [12] in Equation (232) which becomes upon transformation,

$$u(t) = \frac{\dot{y}(t)}{k(t) y(t)} \quad , \tag{247}$$

which becomes

$$\ddot{y}(t) = \frac{\dot{k}(t)}{k(t)} \dot{y}(t) + k(t)\, g(t)\, y(t) \quad . \tag{248}$$

The procedure would be similar to that in the treatment of the forced damped oscillator, and will not be pursued here.

Equations (241) or (245) would be of use only in a one-degree-of-freedom simulation, say rising or falling. In general,

$$\dot{u} = -kVu + g \tag{249}$$

$$\dot{v} = -kVv + h \tag{250}$$

$$\dot{w} = -kVw + i \tag{251}$$

where

$$V = (u^2 + v^2 + w^2)^{1/2} \quad . \tag{252}$$

In body-fixed coordinates, $u \simeq V$,

$$\dot{u} = -k \left[1 + \left(\frac{v}{u}\right)^2 + \left(\frac{w}{u}\right)^2 \right]^{\frac{1}{2}} u^2 + g \quad , \tag{253}$$

$$\dot{v} = -k \left[1 + \left(\frac{v}{u}\right)^2 + \left(\frac{w}{u}\right)^2 \right]^{\frac{1}{2}} uv + h \quad , \tag{254}$$

$$\dot{w} = -k \left[1 + \left(\frac{v}{u}\right)^2 + \left(\frac{w}{u}\right)^2 \right]^{\frac{1}{2}} uw + i \quad . \tag{255}$$

Let

$$k' = k(1 + \alpha^2)^{\frac{1}{2}} \tag{256}$$

where

$$\alpha = \left[\left(\frac{v}{u}\right)^2 + \left(\frac{w}{u}\right)^2 \right]^{\frac{1}{2}} \quad , \tag{257}$$

is the total angle-of-attack. Then Equations (249), (250), and (251) become

$$\dot{u} = -k' u^2 + g \tag{258}$$

$$\dot{v} = -k'uv + h \tag{259}$$

$$\dot{w} = -k'uw + i \quad . \tag{260}$$

Equation (258) may be solved using Equations (241) or (245). The recurrence for Equation (259) would be

$$v_n \simeq \left(\frac{1 - \frac{T}{2} k'_{n-1} u_{n-1}}{1 + \frac{T}{2} k'_n u_n} \right) v_{n-1} + \frac{T}{2} \left(\frac{h_n + h_{n-1}}{1 + \frac{T}{2} k'_n u_n} \right) \quad . \tag{261}$$

The recurrence for Equation (260) would be the same as for Equation (259); that is, the obvious substitutions are made in Equation (261). The rate equations would be handled in a similar manner.

VI. CONCLUSIONS AND RECOMMENDATIONS

Those who didn't know have probably guessed that a potpourri is a "dish of severall meates boyled and stued together."* Some might be of the opinion it should have been allowed to cook a little longer, but this report was not intended as a final summary; it was intended to document the results of an initial investigation and suggest avenues for further development. They are as follows:

a) Place z-transforms on a firm foundation using distribution theory. This has already been done with Laplace and Fourier transforms [13]. The intimate relation between z-transforms and the Dirac delta is suggested in Appendix A. Such a foundation would allow extensions and further development of z-transforms.

b) The effects of tuning for other inputs and transfer functions require analysis. The times when modified z-transforms and/or tunable convolution are advantageous and in what combination they are advantageous require study. This would include the use of higher order holds. The first-order hold leads to Simpson's rule, Equation (59).

Ultimately, it is hoped that there would be some unification between z-transforms, classical numerical methods, and distribution theory. And catastrophe theory...

---

*Also, a literary production composed of unconnected parts.

REFERENCES

1. Smith, J. M., Mathematical Modeling and Digital Simulation for Engineers and Scientists, New York: John Wiley and Sons, 1977.

2. Smith, J. M., "Recent Developments in Numerical Integration," Transactions of ASME, Journal of Dynamics Systems, Measurement and Control, March 1974, pp. 61-70.

3. Jury, E. I., Theory and Application of the z-Transform Method, New York: John Wiley and Sons, 1966.

4. Rosko, J. S., Digital Simulation of Physical Systems, Reading, Pennsylvania: Addison-Wesley Publishing Company, 1972.

5. Cadzow, J. A. and Martens, H. R., Discrete-Time and Computer Control Systems, Englewood Cliffs, New Jersey: Prentice-Hall Incorporated, 1970.

6. Smith, J. M., "Zero Order T-Integration and Its Relation to the Mean Value Theorem," Procedures of the Sixth Annual Pittsburgh Conference, Modeling and Simulation, Pittsburgh: Instrument Society of America, 1975, pp. 499-504.

7. Halijak, C. A., Sampled-Data Systems and Generating Functions, NASA CR-61349, March 1971.

8. Halijak, C. A., "Digital Approximation of Solutions of Differential Equations using Trapezoidal Convolution," Bendix Corporation, Bendix Systems Division, 26 August 1960, ITM-6426.

9. Hamming, R. W., Numerical Methods for Scientists and Engineers, New York: McGraw-Hill Book Company, 1973.

10. Baker, G. A., Jr., Essentials of Padé Approximants, New York: Academic Press, 1975.

11. Dickson, R. E., Motion of an Accelerated Body in a Medium where the Resistance Varies as the Square of the Velocity, US Army Missile Command, Redstone Arsenal, Alabama, May 1971, Report No. RD-TM-71-2.

12. Davis, H. T., Introduction to Nonlinear Differential and Integral Equations, Atomic Energy Commision, Government Printing Office, Washington, D.C., 1960 (out of print; see Dover for reprint).

13. Kecs, W. and Teoderescu, P. P., Application of the Theory of Distributions in Mechanics, Tunbridge Wells, Kent: Abacus Press, 1974.

Appendix A. Z-TRANSFORMS AND OTHER DEFINITIONS AND RELATIONSHIPS

Laplace transform:

$$\mathcal{L}(f(t)) \equiv \int_{0}^{\infty} f(t)e^{-st}\, dt \qquad \text{(A-1)}$$

$$\equiv f(s) \quad . \qquad \text{(A-2)}$$

Convolution:

$$f(t) * g(t) \equiv \int_{0}^{t} f(\tau)\, g(t - \tau)d\tau \qquad \text{(A-3)}$$

$$\mathcal{L}\,(f(t) * g(t)) = f(s)\, g(s) \quad . \qquad \text{(A-4)}$$

The Dirac delta, $\delta(t)$, is the unity element in convolution.

Cauchy Product of Power Series:

$$\left(\sum_{n=0}^{\infty} f_n\right)\left(\sum_{n=0}^{\infty} g_n\right) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} f_n\, g_{n-k} \quad . \qquad \text{(A-5)}$$

Sampling with Dirac delta distribution:

$$f(nT) = \int_{-\infty}^{\infty} \delta(t - nT)\, f(t)\, dt \quad . \qquad \text{(A-6)}$$

z-transform:

$$Z(\mathcal{L}(f(t))) = \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} \delta(t - nT) f(t) e^{-st}\, dt \quad . \qquad \text{(A-7)}$$

It is noted that $f(t)e^{-st}$ is being sampled, then summed. From the properties of the Dirac delta,

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$$Z\left(\mathcal{L}\left[f(t)\right]\right) = \int_{-\infty}^{\infty} f(t)e^{-st} \sum_{n=0}^{\infty} \delta(t - nT)dt \qquad \text{(A-8)}$$

$$= \sum_{n=0}^{\infty} f\,(nT)e^{-nsT} \qquad \text{(A-9)}$$

$$= \sum_{n=0}^{\infty} f(nT)z^{n} \qquad \text{(A-10)}$$

$$= f(z) \qquad \text{(A-11)}$$

Equation (A-10) seems to arise in the literature like Venus full grown from the foam, or is it like Athena from the forehead of Zeus?

Modified z-transform:

$$\sum_{n=0}^{\infty} \int_{-\infty}^{\infty} \delta(t - (n + \eta)T)f(t)e^{-st}\, dt \quad , 0 \le \eta < 1 \qquad \text{(A-12)}$$

$$f(z) = \sum_{n=0}^{\infty} f(nT + \eta T)e^{-(n+\eta)sT} \qquad \text{(A-13)}$$

$$= e^{-\eta sT} \sum_{n=0}^{\infty} f(nT + \eta T)e^{-nsT} \qquad \text{(A-14)}$$

but

$$f(z, \eta) \equiv \sum_{n=0}^{\infty} f(nT + \eta T)e^{-nsT} \quad ; \qquad \text{(A-15)}$$

therefore,

$$z^{n} f(z) = f(z, \eta) \qquad \text{(A-16)}$$

where z^{n} is a fractional shift.

Shifting theorem:

$$\sum_{n=0}^{\infty} \int_{-\infty}^{\infty} \delta(t - (n + m)T)f(t)e^{-st}\, dt \quad , 0 < m \tag{A-17}$$

$$\sum_{n=0}^{\infty} f(nT + mT)e^{-(n+m)sT} \tag{A-18}$$

$$= e^{-msT} \sum_{n=0}^{\infty} f(nT + mT)e^{-nsT} \tag{A-19}$$

$$= e^{-msT}\, f(z,m) \tag{A-20}$$

$$= z^{m}\, f(z,m) \tag{A-21}$$

$$= f(z) - \sum_{n=0}^{m-1} f(nT)z^{n} \quad ; \tag{A-22}$$

therefore,

$$f(z,m) = z^{-m} \left[f(z) - \sum_{n=0}^{m-1} f(nT)z^{n} \right] \tag{A-23}$$

or

$$z^{-m}\, f(z) = f(z,m) + z^{-m} \sum_{n=0}^{m-1} f(nT)z^{n} \quad , \tag{A-24}$$

the shifting theorem.

Shifting the other way,

$$\sum_{n=0}^{\infty} \int_{-\infty}^{\infty} \delta(t - (n - m)T)f(t)e^{-st}\, dt \quad , 0 < m \quad , \tag{A-25}$$

$$= \sum_{n=0}^{\infty} f(nT - mT)e^{-(n-m)sT} \tag{A-26}$$

$$= e^{msT} \sum_{n=0}^{\infty} f(nT - mT)e^{-nsT} \tag{A-27}$$

$$= z^{-m} f(z,-m) \tag{A-28}$$

$$= f(z) + \sum_{k=-m}^{-1} f(kT)e^{-ksT} \quad ; \tag{A-29}$$

therefore,

$$f(z,-m) = z^{m} \left[f(z) + \sum_{k=-m}^{-1} f(kT)z^{k} \right] \tag{A-30}$$

or

$$z^{m} f(z) = f(z,-m) - z^{m} \sum_{k=-m}^{-1} f(kT)\, z^{k} \tag{A-31}$$

$$= f(z,-m) - \sum_{n=0}^{m-1} f(nT - mT)z^{n} \tag{A-32}$$

Just because $f(nT)$, $n < 0$, may not be known does not mean it is zero.

Appendix B. TRANSFORM TABLE FOR SELECTED FUNCTIONS* AND DISTRIBUTIONS

*Healy, M., Tables of Laplace, Heaviside, Fourier and Z-transforms, London: W. and R. Chambers Ltd., 1967. Cadzow, J. A., Discrete-Time Systems, Englewood Cliffs, New Jersey: Prentice Hall, 1973.

f_o	$f(t)$	$f(s)$	$f(z)$	$f(z,\eta)$
1	1	$\frac{1}{s}$	$\frac{1}{1 - z}$	$\frac{1}{1 - z}$
0	t	$\frac{1}{s^2}$	$\frac{Tz}{(1 - z)^2}$	$\frac{T\eta + T(1 - \eta)z}{(1 - z)^2}$
0	t^2	$\frac{2!}{s^3}$	$\frac{T^2 z(1 + z)}{(1 - z)^2}$	$\frac{T^2[\eta^2 + (1 - 2\eta - 2\eta^2)z + (1 + 2\eta - \eta^2)z^2]}{(1 - z)^3}$
0	t^3	$\frac{3!}{s^4}$	$\frac{T^3 z(1 + 4z + z^2)}{(1 - z)^4}$	
1	e^{-at}	$\frac{1}{s + a}$	$\frac{1}{1 - z e^{-aT}}$	$\frac{e^{-\eta aT}}{1 - z e^{-aT}}$
0	$t e^{-at}$	$\frac{1}{(s + a)^2}$	$\frac{Tz e^{-aT}}{(i - z e^{-aT})^2}$	$\frac{T e^{-\eta aT} [z e^{-aT} + \eta(1 - z e^{-aT})]}{(1 - z e^{-aT})^2}$
0	$1 - e^{-at}$	$\frac{a}{s(s + a)}$	$\frac{(1 - e^{-aT})z}{(1 - z)(1 - z e^{-aT}}$	$\frac{(1 - e^{-\eta aT}) - (e^{-aT} - e^{-\eta aT})z}{(1 - z)(1 - z e^{-aT})}$

f_o	f(t)	f(s)	f(z)	$f(z,\eta)$
0	$at + e^{-at} - 1$	$\frac{a^2}{s^2(s+a)}$	$\frac{(aT + e^{-aT} - 1)z + (1 - e^{-aT} - aT\,e^{-aT})z^2}{(1-z)^2\,(1 - z\,e^{-aT})}$	$\frac{aT\,z}{(1-z)^2} + \frac{(a\eta - 1)}{(1-z)} + \frac{a\,e^{-\eta aT}}{(1 - z\,e^{-aT})}$
0	$1 - (1 + at)e^{-at}$	$\frac{a^2}{s(s+a)^2}$	$\frac{1}{1-z} - \frac{1 + z\,e^{-aT}(aT - 1)}{(1 - z\,e^{-aT})}$	$\frac{1}{1-z} - e^{-\eta aT}\left[\frac{1 + \eta aT}{1 - z\,e^{-aT}} + \frac{z\,aTe^{-aT}}{(1 - z\,e^{-aT})^2}\right]$
1	$\cos \omega t$	$\frac{s}{s^2 + \omega^2}$	$\frac{1 - z\cos\omega T}{1 - 2z\cos\omega T + z^2}$	$\frac{\cos\eta\omega T - z\cos(1-\eta)\omega T}{1 - z\cos\omega T + z^2}$
0	$\sin \omega t$	$\frac{\omega}{s^2 + \omega^2}$	$\frac{z\sin\omega T}{1 - 2z\cos\omega T + z^2}$	$\frac{\sin\eta\omega T + z\sin(1-\eta)\omega T}{1 - z\cos\omega T + z^2}$
1	$e^{-at}\cos\omega t$	$\frac{s+a}{(s+a)^2 + \omega^2}$	$\frac{1 - z\,e^{-aT}\cos\omega T}{1 - 2z\,e^{-aT}\cos\omega T + e^{-2aT}z^2}$	$\frac{e^{-\eta aT}[\cos\eta\omega T - z\,e^{-aT}\cos(1-\eta)\omega T]}{1 - 2z\,e^{-aT}\cos\omega T + e^{-2aT}z^2}$
0	$e^{-at}\sin\omega t$	$\frac{\omega}{(s+a)^2 + \omega^2}$	$\frac{z\,e^{-aT}\sin\omega T}{1 - 2z\,e^{-aT}\cos\omega T + e^{-2aT}z^2}$	$\frac{e^{-\eta aT}[\sin\eta\omega T + z\,e^{-aT}\sin(1-\eta)\omega T]}{1 - z\,e^{-aT}\cos\omega T + e^{-2aT}z^2}$
1	$\cosh\omega t$	$\frac{s}{s^2 - \omega^2}$	$\frac{1 - z\cosh\omega T}{1 - 2z\cosh\omega T + z^2}$	$\frac{\cosh\eta\omega T - z\cosh(1-\eta)\omega T}{1 - 2z\cosh\omega T + z^2}$
0	$\sinh\omega t$	$\frac{\omega}{s^2 - \omega^2}$	$\frac{z\sinh\omega T}{1 - 2z\cosh\omega T + z^2}$	$\frac{\sinh\eta\omega T + z\sinh(1-\eta)\omega T}{1 - 2z\cosh\omega T + z^2}$
1	$e^{-at}\cosh\omega t$	$\frac{s+a}{(s+a)^2 - \omega^2}$	$\frac{z\sinh\omega T}{1 - 2z\cosh\omega T + z^2}$	$\frac{e^{-\eta aT}[\cosh\eta\omega T - z\,e^{-aT}\cosh(1-\eta)\omega T]}{1 - 2z\,e^{-aT}\cosh\omega T + z^2\,e^{-2aT}}$

f_o	$f(t)$	$f(s)$	$f(z)$	$f(z,\eta)$
0	$e^{-at} \sinh \omega t$	$\frac{\omega}{(s + a)^2 - \omega^2}$	$\frac{z\, e^{-aT} \sinh \omega T}{1 - 2\, z\, e^{-aT} \cosh \omega T + z^2 e^{-2aT}}$	$\frac{e^{-\eta aT}[\sinh \eta\omega T + z\, e^{-aT} \sinh(1 - \eta)\omega T]}{1 - z\, e^{-aT} \cosh \omega T + z^2 e^{-2aT}}$
$\cos\phi$	$\cos(\omega t + \phi)$	$\frac{s \cos\phi + \omega \sin\phi}{s^2 + \omega^2}$	$\frac{\cos\phi - z \cos(\omega T - \phi)}{1 - 2\, z \cos \omega T + z^2}$	$\frac{\cos(\eta\omega T + \phi) - z \cos(1 - \eta)\omega T - \phi)}{1 - 2\, z \cos \omega T + z^2}$
$\sin\phi$	$\sin(\omega t + \phi)$	$\frac{\omega \cos\phi + s \sin\phi}{s^2 + \omega^2}$	$\frac{\sin\phi + z \sin(\omega T + \phi)}{1 - 2\, z \cos \omega T + z^2}$	$\frac{\sin(\omega T + \phi) + z \sin((1 - \eta)\omega T + \phi)}{1 - 2\, z \cos \omega T + z^2}$
$\delta(o)$	$\delta(t)$	1	$\delta(o)$	$z^{-m}\delta(o)$
$\delta'(o)$	$\delta'(t)$	s	$\delta'(o)$	$z^{-m}\delta'(o)$
$\delta^{(n)}(o)$	$\delta^{(n)}(t)$	s^n	$\delta^{(n)}(o)$	$z^{-m}\delta^{(n)}(o)$
$\delta(nT)$	$\delta(t - nT)$	e^{-nsT}	$z^n\delta(o)$	$z^{n-m}\delta(o)$

Appendix C. DIRAC DELTA DISTRIBUTION*

$$\int_{-\infty}^{\infty} f(t)\ \delta(t - nT)dt = f(nT) \tag{C-1}$$

where f(t) is a well-defined function at t = nT.

$$\delta(t) = \delta(-t) \tag{C-2}$$

$$\delta(at) = \frac{1}{|a|}\ \delta(t) \tag{C-3}$$

$$\delta[g(t)] = \sum_{n} \frac{1}{|g'(nT)|}\ \delta(t - nT), \quad \begin{matrix} g(nT) = 0 \\ g'(nT) \neq 0 \end{matrix} \tag{C-4}$$

$$t\ \delta(t) = 0 \tag{C-5}$$

$$f(t)\delta(t - nT) = f(nT)\delta(t - nT) \tag{C-6}$$

$$\int \delta(t - \tau)\delta(t - nT)dt = \delta(\tau - nT) \tag{C-7}$$

$$\int_{-\infty}^{\infty} \overset{(m)}{\delta}(t)\ f(t) = (-1)^m\ \overset{(m)}{f}(o) \tag{C-8}$$

$$\int_{-\infty}^{\infty} \overset{(1)}{\delta}(o)\ e^{-st}\ dt = s \tag{C-9}$$

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*Dirac, P. A. M., The Principles of Quantum Mechanics, London: Oxford University Press, 1958, Messiah, A., Quantum Mechanics, New York: John Wiley and Sons, 1964.

Appendix D. PAST VALUES FOR DIFFERENCE EQUATIONS SOLVING DIFFERENTIAL EQUATIONS WITH INITIAL CONDITIONS

As was seen in the sample problems in Section V, the order of the plant (and the forcing function) determine the number of start-up steps before the recurrence may be applied. In a simulation containing different order differential equations, the recurrences would start on different steps, which could add complexity to the simulation.

To start all recurrences, regardless of their order at Step 1 (n = 1), in general, values before n = o will be required. First, the required number of steps must be solved backwards in time before proceeding forward in time with the recurrence at Step 1. The differential equation could also be solved backwards in time (odd derivatives and functions change sign, etc., and the startup steps would be obtained for computing backwards as well as the backwards recurrence, which is not required.

If the forward start-up steps have already been determined, the backwards ones may be determined by substituting -T for T and -n for n. This would also apply for the recurrence. For example, Equations (161), (160), and (159) would become, respectively,

$$x_{-n} = 2\,x_{-(n-1)} - x_{-(n-2)} + T^2\,\ddot{x}_{-(n-1)},\ -n < -1$$

$$= 2\,x_{-n+1} - x_{-n+2} + T^2\,\ddot{x}_{-n+1},\ n > 1 \qquad \text{(D-1)}$$

$$x_{-1} = x_o - T\,\dot{x}_o + \frac{T^2}{2}\,\ddot{x}_o,\ n = -1 \quad , \qquad \text{(D-2)}$$

and

$$x_o = x_o \quad ,\ n = o \quad . \qquad \text{(D-3)}$$

With Equations (D-2) and (D-3), Equation (161) may then be written as:

$$x_n = 2\,x_{n-1} - x_{n-2} + T^2\,\ddot{x}_{n-1},\ n \geq 1 \quad . \qquad \text{(D-4)}$$

Because the equations were solved backwards, the past values required for starting the recurrence at Step 1 (n = 1) are not necessarily the past values of the actual system. The forcing function may not have been present before time zero (n = o); nevertheless, it must be taken into consideration to start the recurrence at Step 1.

A numerical example will illustrate. For $x_o = 1$, $\dot{x}_o = 1$, $\ddot{x}_o = 2$, and $T = 1$, Equations (159), (160), and (161) become

$$x_o = 1$$

$$x_1 = 1 + 1 \times 1 + \frac{1}{2} \times 2 = 3$$

$$x_2 = 2 \times 3 - 1 + 1 \times 2 = 7 \quad .$$

Using Equations (D-2), (D-3), and (D-4) twice,

$$x_{-1} = 1 - 1 + \frac{1}{2} \times 2 = 1$$

$$x_o = 1$$

$$x_1 = 2 \times 1 - 1 + 1 \times 2 = 3$$

$$x_2 = 2 \times 3 - 1 + 1 \times 2 = 7 \quad .$$

The free running oscillator, Equations (203), (204), and (205), noting that cosine is an even function and sine an odd function, would become

$$x_{-1} = x_o (\cos \omega T) - \dot{x}_o (\sin \omega T) \quad , n = -1 \quad , \tag{D-5}$$

$$x_o = x_o \quad , n = o \quad , \tag{D-6}$$

and

$$x_n = (2 \cos \omega T)\, x_{n-1} - x_{n-2}, \quad n \geq 1 \quad . \tag{D-7}$$

In this way a "preprogram" could convert initial conditions for the differential equations into the required past values for the difference equations at Step 1.

There is considerable confusion on this point in the litueratureand caution is advised.

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